Time reparametrization symmetry in spin-glass models

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We study the long-time aging dynamics of spin-glass models with two-spin interactions by performing a renormalization group (RG) transformation on the time variable in the nonequilibrium dynamical generating functional. We obtain the RG equations and find that the flow converges to an exact fixed point. We show that this fixed point is invariant under reparametrizations of the time variable. This continuous symmetry is broken, as evidenced by the fact that the observed correlations and responses are not invariant under it. We argue that this gives rise to the presence of Goldstone modes, and that those Goldstone modes shape the behavior of fluctuations in the nonequilibrium dynamics.

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I. INTRODUCTION

Glassy materials are characterized by very slow dynamics, associated with a dramatic slowdown of molecular relaxation in structural glasses, and with a dramatic slowdown of spin relaxation in spin glasses. This slowdown of the dynamics has been captured in great part by the results obtained by dynamical mean-field theories. In the case of supercooled liquids, the mean-field mode-coupling approach[1] has been successful in describing some of the features of the relaxation. In the case of spin glasses, a dynamical mean-field theory based on Langevin dynamics for the spins, examined within a functional integral formulation of the Martin-Siggia-Rose approach,[2–5] has been used to study the long-time relaxation. The dynamical mean-field theory of spin glasses has successfully captured[3–5] some unusual properties of the spin dynamics, associated with the lack of equilibration, including the presence of physical aging and the breakdown of the equilibrium fluctuation-dissipation relations.

However, mean-field theories do not allow direct access to a description of the fluctuations in the dynamics. It turns out that fluctuations in the dynamics of glassy systems can in fact be rather strong, as it has been underlined by the discovery of *dynamical heterogeneities*.[6,7] Dynamical heterogeneities are nanometer-scale regions of molecules rearranging cooperatively at very different rates compared to the bulk. Recent studies of material systems near their glass transitions have uncovered substantial experimental[8–17] and simulational[18–22] evidence for their presence. Various attempts at theoretically addressing these strong fluctuations have been made, involving, among others, the ideas of dynamic facilitation,[23–27] the presence of a "random first-order phase transition,"[28–30] or the use of diagrammatic methods to carefully reanalyze and extend mode-coupling theory.[31,32] However, a detailed theory that explains the dynamical heterogeneities remains elusive.[33]

Recently, a theoretical framework for the study of fluctuations in the nonequilibrium dynamics of glassy systems has been proposed,[34–37] which is based on the presence of a Goldstone mode associated with a symmetry under continuous reparametrizations of the time variable. It was argued there that the presence of this symmetry could provide an explanation for many of the dynamical heterogeneity effects observed in various glassy systems. In Ref. 34, a sketch of a proof for the presence of this symmetry was presented. Earlier work had uncovered the presence of a restricted version of this symmetry for the mean-field dynamical equations of some infinite-range spin-glass models.[3–5,38]

In the present work, we present a detailed proof of the presence of this symmetry under continuous reparametrizations of the time variable, for the long-time dynamics of a generic spin-glass model with two-spin interactions. The proof is based on using the renormalization group (RG) to extract the long-time behavior of the theory. It is somewhat unusual in the sense that we coarse grain *time differences* and not positions. In other words, the degrees of freedom that are "integrated over" are the ones associated with the "fast" dynamics, where by "fast" we mean fast in time, and not necessarily in space.

Although involved in some of its details, our procedure is conceptually simple. We consider a model for a set of soft spins on a lattice, which contains only two-spin interactions, with a zero-mean uncorrelated Gaussian distribution for the spin couplings. We assume a Langevin-type dynamics for the spins with a noise term whose amplitude is controlled by the temperature of the environment. We use the functional integral formulation of the Martin-Siggia-Rose approach to describe the Langevin dynamics. We set up the calculation by writing the generating functional for the spin correlations and responses, and find that this generating functional can be written in terms of a functional integral over an auxiliary field that depends on two times. We set up the renormalization group procedure by defining a cutoff τ_0 for the time differences. We increase the cutoff slightly and integrate over all values of the auxiliary field that correspond to time differences smaller than this slightly increased cutoff. This integral is actually a Gaussian integral that can be performed exactly. After integrating over the fast variables, we rescale all times in such a way that the cutoff goes back to its original value τ_0. We find that the RG flow converges to a fixed point, which defines the fixed-point generating functional. Finally, we consider a smooth and monotonously increasing but otherwise arbitrary reparametrization of the time variable $t \to s(t)$, which induces a transformation of the sources for the generating functional. We compute the value of the fixed-point generating functional for those transformed values of the sources, and show that it is the same as for the original values of the sources. In other words, the reparametrization of the time variable leaves the fixed-point generating functional invariant.

The rest of the paper is organized as follows: in Sec. II we introduce and briefly discuss the spin model, the Martin-Siggia-Rose formalism for the Langevin spin dynamics and the assumptions about the nature of the random couplings; in Sec. III we obtain an explicit form for the disorder-averaged Martin-Siggia-Rose generating functional, which contains the above-mentioned auxiliary fields that play a central role in the formulation of the renormalization group; in Sec. IV we introduce our renormalization group procedure, associated with coarse graining the *time differences*, derive the flow equations for the parameters of the action, and find the fixed point to which the RG flows; in Sec. V we derive the central result of this work, i.e., we show that the fixed-point generating functional is invariant under reparametrizations of the time variable in the sources; and in Sec. VI we discuss the physical consequences expected from the presence of this symmetry, which have already been observed in numerical simulations of spin glasses and structural glasses, and can also be tested for in confocal microscopy experiments in colloidal glasses. Finally, in Sec. VII we summarize our results.

II. MODEL

We consider a spin-glass Hamiltonian containing only two-spin interactions,

$$H_0 = \frac{1}{2}\sum_{r,r'} J_{rr'}\phi_r\phi_{r'} + \sum_r W(\phi_r). \tag{1}$$

Here the indices r and r' label the N possible positions in the (discrete) lattice, ϕ_r are soft spin variables, $J_{rr'}$ are the spin coupling constants (satisfying $J_{rr'}=J_{r'r}$ and $J_{rr'}=0$ for $r=r'$), and the one-spin potential $W(\phi)$ is chosen to control the magnitude of the spin variables. We assume that the potential $W(\phi)$ is real, even, and analytic at $\phi=0$, i.e.,

$$W(\phi) = \sum_{p=0}^{\infty} w_p\phi^{2p}, \tag{2}$$

with $w_p=w_p^*$ $\forall p$. For example, for the potential $W(\phi)=\frac{\lambda}{4}(1-\phi^2)^2$, the coefficients are $w_0=\frac{\lambda}{4}$, $w_1=-\frac{\lambda}{2}$, $w_2=\frac{\lambda}{4}$, and $w_p=0$ $\forall p>2$.

The Langevin equation for the spin variables for a given realization $\xi_r(t)$ of the noise reads

$$\frac{\partial\phi_r}{\partial t} = -\frac{\partial H}{\partial\phi_r} + \xi_r(t). \tag{3}$$

We assume, as usual, that the noise is Gaussian distributed and uncorrelated, with a variance that defines the temperature T of the heat reservoir,

$$\langle\xi_r(t_1)\xi_{r'}(t_2)\rangle = 2T\delta_{r,r'}\delta(t_1-t_2), \tag{4}$$

where the angle brackets $\langle\cdots\rangle$ indicate an average over the noise distribution.

We compute the derivatives

$$\frac{\partial H_0}{\partial\phi_r} = \sum_{r'} J_{rr'}\phi_{r'} + W'(\phi_r), \tag{5}$$

where we have used that for all r, $J_{rr}=0$.

Then the Martin-Siggia-Rose generating functional,[39] averaged over the realizations of the noise, and incorporating the sources, reads

$$\langle Z[\ell,h]\rangle = \int \mathcal{D}\phi\mathcal{D}\hat{\phi}\mathcal{D}\hat{\varphi}\,\exp\left\{L[\phi,\hat{\phi}] + \sum_r\int_{t_0}^{t_f} dt[\ell_r(t)\phi_r(t) + ih_r(t)\hat{\phi}_r(t)] + i\sum_r\hat{\varphi}_r[\phi_r(t_0)-\varphi_r]\right\}, \tag{6}$$

where the notation $\langle(\cdots)\rangle$ indicates the average over the realizations of the noise. We are considering the time evolution between times t_0 and t_f of the spins $\phi_r(t)$, with initial conditions given by the φ_r, i.e., $\forall r$: $\phi_r(t_0)=\varphi_r$, and the action is given (in general) by

$$L[\phi,\hat{\phi}] = -i\sum_r\int_{t_0}^{t_f} dt\hat{\phi}_r(t)\left(\frac{\partial\phi_r(t)}{\partial t} + \left.\frac{\partial H_0}{\partial\phi_r}\right|_{\{\phi\}} - iT\hat{\phi}_r(t)\right). \tag{7}$$

In our case, by using Eq. (5) we obtain

$$L[\phi,\hat{\phi}] = -i\sum_r\int_{t_0}^{t_f} dt\hat{\phi}_r(t) \times\left(\frac{\partial\phi_r(t)}{\partial t} + \sum_{r'}J_{rr'}\phi_{r'} + W'(\phi_r) - iT\hat{\phi}_r(t)\right). \tag{8}$$

We assume that the disorder is given by an uncorrelated, zero-mean, Gaussian distribution for the couplings, i.e.,

$$\mathcal{P}\{J\} = \prod_{r<r'}\left(\frac{\exp(-J_{rr'}^2/4K_{rr'})}{(4\pi K_{rr'})^{1/2}}\right). \tag{9}$$

Here the connectivity matrix $2K_{rr'}=\overline{J_{rr'}^2}$ defines the variances of the random couplings, with the notation $\overline{(\cdots)}$ denoting an average over the disorder. The connectivity matrix $K_{rr'}$ encodes the properties of the model. For example, in the case of the Edwards-Anderson model, $K_{rr'}=K>0$ for r, r' nearest neighbors and is zero otherwise.

The functional $\overline{\langle Z[\ell,h]\rangle}$ allows the direct computation of measurable quantities: expectation values, correlations, and responses. The expectation values and the p-point correlation functions of the field $\phi_r(t)$ are calculated by taking derivatives of $\overline{\langle Z[\ell,h]\rangle}$ with respect to the source ℓ coupled to ϕ (Ref. 39):

$$\overline{\langle \phi_r(t)\rangle} = \left.\frac{\delta \overline{\langle Z[\ell,h]\rangle}}{\delta \ell_r(t)}\right|_{\ell=0,h=0}, \tag{10}$$

$$\begin{aligned} C_p(r_1,t_1;r_2,t_2;\cdots;r_p,t_p) &\equiv \overline{\langle \phi_{r_1}(t_1)\phi_{r_2}(t_2)\cdots\phi_{r_p}(t_p)\rangle} \\ &= \frac{\delta^{(p)}\overline{\langle Z[\ell,h]\rangle}}{\delta\ell_{r_1}(t)\ell_{r_2}(t_2)\cdots\ell_{r_p}(t_p)}. \end{aligned} \tag{11}$$

Here we have used the fact that the generating functional $\overline{\langle Z[\ell,h]\rangle}$ reduces to unity for zero sources; i.e., it satisfies the condition $\overline{\langle Z[\ell=0,h=0]\rangle}=1$.

The effect of (possibly time-dependent) external fields $H=H_0-\Sigma_r h_r(t)\phi_r(t)$ can also be probed by computing response functions,

$$\begin{aligned} R(r,t|r',t') &\equiv \frac{\delta\overline{\langle\phi_r(t)\rangle}}{\delta h_{r'}(t')} = \left.\frac{\delta\overline{\langle Z[\ell,h]\rangle}}{\delta\ell_r(t)\,\delta h_{r'}(t')}\right|_{\ell=0,h=0} \\ &= i\overline{\langle\phi_r(t)\hat{\phi}_{r'}(t')\rangle}, \end{aligned}$$

$$\chi(r,t|r',t') \equiv \int_{t'}^{t} dt'' R(r,t|r',t''). \tag{12}$$

Here $R(r,t|r',t')$ represents the response to an external field only present at time t', i.e., a "delta function in time," and the integrated response $\chi(r,t|r',t')$ corresponds to a "step field," i.e., an external field that is "turned on" at time t' and "stays on" until the time t when the spin is measured. An important property of response functions is that, by causality, the response $R(r,t|r',t')$ is zero for $t'>t$. Expectation values, correlations, and responses for *one disorder realization* can also be computed by formulas that differ from Eqs. (11) and (12) only in that all disorder averaging is removed.

III. DISORDER-AVERAGED GENERATING FUNCTIONAL

Once the distribution of the couplings is defined, we can average the disorder-dependent exponential in the action,

$$\begin{aligned} z_0[\hat{\phi},\phi] &\equiv \overline{\exp\left[-i\int_{t_0}^{t_f} dt \sum_{r,r'} J_{rr'}\hat{\phi}_r(t)\phi_{r'}(t)\right]} \\ &= \prod_{r<r'}\int dJ_{rr'}\frac{\exp(-J_{rr'}^2/4K_{rr'})}{(4\pi K_{rr'})^{1/2}}\exp\left(J_{rr'}\left\{-i\int_{t_0}^{t_f} dt[\hat{\phi}_r(t)\phi_{r'}(t)+\hat{\phi}_{r'}(t)\phi_r(t)]\right\}\right) \\ &= \exp\left\{-\sum_{r,r'}\frac{K_{rr'}}{2}\int_{t_0}^{t_f} dt_1 dt_2[\hat{\phi}_r(t_1)\phi_{r'}(t_1)\hat{\phi}_r(t_2)\phi_{r'}(t_2)+\hat{\phi}_{r'}(t_1)\phi_r(t_1)\hat{\phi}_r(t_2)\phi_{r'}(t_2)\right. \\ &\quad \left. +\hat{\phi}_r(t_1)\phi_{r'}(t_1)\hat{\phi}_{r'}(t_2)\phi_r(t_2)+\hat{\phi}_{r'}(t_1)\phi_r(t_1)\hat{\phi}_{r'}(t_2)\phi_r(t_2)]\right\}. \end{aligned} \tag{13}$$

We now define the notations $\phi_r^0(t)\equiv\hat{\phi}_r(t)$, $\phi_r^1(t)\equiv\phi_r(t)$, $\bar{0}\equiv 1$, and $\bar{1}\equiv 0$, which allow us to write

$$z_0[\hat{\phi},\phi] = \exp\left[-\frac{1}{2}\sum_{r,r'}K_{rr'}\int_{t_0}^{t_f} dt_1 dt_2 \sum_{a,c=0}^{1} \phi_r^a(t_1)\phi_r^c(t_2)\phi_{r'}^{\bar{a}}(t_1)\phi_{r'}^{\bar{c}}(t_2)\right]. \tag{14}$$

Here we can introduce auxiliary two-time fields $Q_r^{ac}(t_1,t_2)$ by performing a Hubbard-Stratonovich transformation,

$$z_0[\hat{\phi},\phi] = \int \mathcal{D}Q \exp\left(-\frac{1}{2}\sum_{r,r'}M_{rr'}\int_{t_0}^{t_f} dt_1 dt_2\sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2)Q_{r'}^{\bar{a}\bar{c}}(t_1,t_2) + i\sum_r\int_{t_0}^{t_f} dt_1 dt_2 \sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2)\phi_r^a(t_1)\phi_r^c(t_2)\right), \tag{15}$$

where $M_{rr'}$ is the matrix inverse of $K_{rr'}$ and $\int\mathcal{D}Q \equiv \mathcal{N}(M)\int\Pi_{r,a,c}\Pi_{t_1,t_2}dQ_r^{ac}(t_1,t_2)$. Here $\mathcal{N}(M)=\{\det[(2\pi)^{-1}M]\}^{1/2}$ is an M-dependent normalization factor.

We are now in a position to write down the disorder-averaged generating functional for the problem,

$$\mathcal{Z}[\ell,h] \equiv \overline{\langle Z[\ell,h]\rangle} = \int \mathcal{D}Q \exp(-S_K[Q]-S_{nl}[Q,\ell,h]), \tag{16}$$

where

$$S_K[Q] \equiv \frac{1}{2}\sum_{r,r'} M_{rr'} \int_{t_0}^{t_f} dt_1 dt_2 \sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2) Q_{r'}^{\bar{a}\bar{c}}(t_1,t_2), \tag{17}$$

$$S_{nl}[Q,\ell,h] \equiv -\ln \int \mathcal{D}\phi^0 \mathcal{D}\phi^1 \mathcal{D}\hat{\varphi} \exp\Bigg\{ iS_{\text{HS}}[Q,\phi^0,\phi^1] + iS_{\text{spin}}[\phi^0,\phi^1] + iS_{\text{BC}}[\phi^1,\hat{\varphi}] + \sum_r \int_{t_0}^{t_f} dt[\ell_r(t)\phi_r(t) + ih_r(t)\hat{\phi}_r(t)] \Bigg\}, \tag{18}$$

$$S_{\text{HS}}[Q,\phi^0,\phi^1] = \sum_r \int_{t_0}^{t_f} dt_1 dt_2 \sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2) \phi_r^a(t_1) \phi_r^c(t_2), \tag{19}$$

$$S_{\text{spin}}[\phi^0,\phi^1] = -\sum_r \int_{t_0}^{t_f} dt \phi_r^0(t) \left(\frac{\partial \phi_r^1(t)}{\partial t} + W'(\phi_r^1) - iT\phi_r^0(t) \right), \tag{20}$$

$$S_{\text{BC}}[\phi^1,\hat{\varphi}] = \sum_r \hat{\varphi}_r \{\phi_r^1(t_0) - \varphi_r\}. \tag{21}$$

To simplify the algebra, we take from now on the integration limits as $t_0=0$ and $t_f \to \infty$. By combining Eqs. (2) and (20), we write S_{spin} in a way that will allow the RG equations to be put in a simple form,

$$S_{\text{spin}}[\phi^0,\phi^1] = -\sum_r \int_0^\infty dt \Bigg\{ \frac{1}{\Gamma} \phi_r^0(t) \frac{\partial \phi_r^1(t)}{\partial t} + \sum_{a,c=0}^{1} \gamma_{ac}^{(2)} \phi_r^a(t) \phi_r^c(t) + \sum_{p=2}^{\infty} \gamma^{(2p)} \phi_r^0(t) \times [\phi_r^1(t)]^{2p-1} \Bigg\} + \frac{i}{2} \int_0^\infty dt_1 dt_2 \sum_{r,r'} K_{rr'} \times \sum_{a,c=0}^{1} g^{(4)}(t_1-t_2) \phi_r^a(t_1) \phi_r^c(t_2) \phi_{r'}^{\bar{a}}(t_1) \phi_{r'}^{\bar{c}}(t_2). \tag{22}$$

IV. RENORMALIZATION GROUP

We want to introduce an RG transformation on the *time variables*. Since the construction of the RG transformation is a bit unusual, we will explain it in detail. We introduce a short-time cutoff $\tau_0 = 1/\Omega_0$ for the time difference $t_1 - t_2$. This only affects the terms in the action containing an integration over two-time variables, namely, $S_K[Q]$, $S_{\text{HS}}[Q,\phi^0,\phi^1]$, and $S_{\text{spin}}[\phi^0,\phi^1]$. The first two terms take the following form as a starting point for the RG:

$$S_K[Q] = \frac{1}{2}\sum_{r,r'} M_{rr'} \int_{\substack{\tau_0 \le |t_1-t_2| \\ 0 \le t_1,t_2 < \infty}} dt_1 dt_2 \sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2) Q_{r'}^{\bar{a}\bar{c}}(t_1,t_2), \tag{23}$$

$$S_{\text{HS}}[Q,\phi^0,\phi^1] = \sum_r \int_{\substack{\tau_0 \le |t_1-t_2| \\ 0 \le t_1,t_2 < \infty}} dt_1 dt_2 \sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2) \times \phi_r^a(t_1) \phi_r^c(t_2). \tag{24}$$

These terms differ from Eqs. (17) and (19) by the removal of the contributions corresponding to $|t_1 - t_2| < \tau_0$. There are two possible natural assumptions about how this cutoff is implemented: either we assume (i) that the contributions for those time pairs is directly removed from $S_K[Q]$ and $S_{\text{HS}}[Q,\phi^0,\phi^1]$ without any effects on other terms in the action, or (ii) that the Hubbard-Stratonovich transformation performed to obtain Eq. (15) is undone for time pairs $|t_1 - t_2| < \tau_0$. These two alternative assumptions lead to slightly different starting points for the RG flow, but in the end the flow converges to exactly the same fixed point in both cases. This is reassuring, in the sense that we expect the properties of the long-time dynamics not to depend on the cutoff procedure. The initial coefficients for $S_{\text{spin}}[\phi^0,\phi^1]$ in Eq. (22) are given by

$$\Gamma = 1,$$
$$\gamma_{00}^{(2)} = -iT,$$
$$\gamma_{01}^{(2)} = \gamma_{10}^{(2)} = 2w_1,$$
$$\gamma_{11}^{(2)} = 0,$$
$$\gamma^{(2p)} = 2pw_p \quad \forall\, p \ge 2, \tag{25}$$

$$g^{(4)}(t_1 - t_2) = \begin{cases} 0 & \text{for cutoff procedure (i)} \\ \mathcal{C}_{|t_1-t_2|<\tau_0} & \text{for cutoff procedure (ii)}. \end{cases} \tag{26}$$

Here the characteristic function $\mathcal{C}_{\mathcal{P}}$ is defined to be 1 if $\mathcal{P}$ is true and 0 if $\mathcal{P}$ is false.

We now perform an RG transformation on the time variables. We separate the two-time fields Q into fast modes $Q_>$ and slow modes $Q_<$,

$$Q_{>,r}^{ac}(t_1,t_2) \equiv \begin{cases} Q_r^{ac}(t_1,t_2) & \text{for } \tau_0 \le |t_1 - t_2| < b\tau_0 \\ 0 & \text{for } b\tau_0 \le |t_1 - t_2|, \end{cases} \tag{27}$$

$$Q_{<,r}^{ac}(t_1,t_2) \equiv \begin{cases} 0 & \text{for } \tau_0 \le |t_1 - t_2| < b\tau_0 \\ Q_r^{ac}(t_1,t_2) & \text{for } b\tau_0 \le |t_1 - t_2|, \end{cases} \tag{28}$$

with $b > 1$. Clearly, we have

$$Q_r^{ac}(t_1,t_2) = Q_{>,r}^{ac}(t_1,t_2) + Q_{<,r}^{ac}(t_1,t_2), \tag{29}$$

and by inspecting Eq. (23) we find that

$$S_K[Q] = S_K[Q_> + Q_<] = S_K[Q_>] + S_K[Q_<]. \tag{30}$$

As our next step, we integrate over the fast variables $Q_>$ to obtain

$$
\begin{aligned}
S_{\Omega/b}[Q_<] &= -\ln\int \mathcal{D}Q_> \exp\{-S_\Omega[Q_> + Q_<]\} \\
&= F_\Omega + \frac{1}{2}\sum_{r,r'} M_{rr'} \int_{\substack{b\tau_0\le|t_1-t_2| \\ 0\le t_1,t_2<\infty}} dt_1 dt_2 \sum_{a,c=0}^{1} Q^{ac}_{<,r}(t_1,t_2) Q^{\bar{a}\bar{c}}_{<,r'}(t_1,t_2) - \ln\int \mathcal{D}\phi^0\mathcal{D}\phi^1\mathcal{D}\hat{\varphi}\Bigg\{\int \mathcal{D}Q_> \exp\Bigg(-\frac{1}{2}\sum_{r,r'} M_{rr'} \\
&\quad\times \int_{\substack{\tau_0\le|t_1-t_2|<b\tau_0 \\ 0\le t_1,t_2<\infty}} dt_1 dt_2 \sum_{a,c=0}^{1} Q^{ac}_{>,r}(t_1,t_2) Q^{\bar{a}\bar{c}}_{>,r'}(t_1,t_2) + i\sum_r \int_{\substack{\tau_0\le|t_1-t_2|<b\tau_0 \\ 0\le t_1,t_2<\infty}} dt_1 dt_2 \sum_{a,c=0}^{1} Q^{ac}_{>r}(t_1,t_2)\phi^a_r(t_1)\phi^c_r(t_2)\Bigg)\Bigg\} \\
&\quad\times \exp\Bigg(i\sum_r \int_{\substack{b\tau_0\le|t_1-t_2| \\ 0\le t_1,t_2<\infty}} dt_1 dt_2 \sum_{a,c=0}^{1} Q^{ac}_{<,r}(t_1,t_2)\phi^a_r(t_1)\phi^c_r(t_2) + iS_{\text{spin}}[\phi^0,\phi^1] + iS_{\text{BC}}[\phi^1,\hat{\varphi}]\Bigg). \qquad (31)
\end{aligned}
$$

The factor $\{\int \mathcal{D}Q_> \exp\{\cdots\}\}$, which contains the integration over the fast modes $Q_>$, is actually a Gaussian integral, which evaluates to

$$
(\det\{(2\pi)^{-1}K\})^{\mathcal{V}(\tau_0,b)/2} \exp\Bigg(-\frac{1}{2}\sum_{r,r'} K_{rr'} \int_{\substack{\tau_0\le|t_1-t_2|<b\tau_0 \\ 0\le t_1,t_2<\infty}} dt_1 dt_2 \times \sum_{ac=0}^{1} \phi^a_r(t_1)\phi^c_r(t_2)\phi^{\bar{a}}_{r'}(t_1)\phi^{\bar{c}}_{r'}(t_2)\Bigg), \qquad (32)
$$

where $\mathcal{V}(\tau_0,b)$ is proportional to the volume of the two-dimensional (time) region where the condition $\tau_0\le|t_1-t_2| < b\tau_0$ holds. In this expression, the determinant prefactor contributes to the renormalization of the constant term F_Ω, and the argument of the exponential contributes to the renormalization of the function $g^{(4)}(t_1-t_2)$.

We now not only perform the rescaling of the fields $Q^{ac}_{<,r}(t_1,t_2)$ and the time variable, as it would normally be done for an RG procedure, but we also simultaneously rescale the $\phi^a_r(t)$ fields and the sources $\{\ell_r(t),h_r(t)\}$, even though those quantities were not subject to the integration of fast modes,

$$Q^{ac}_{<,r}(bt_1',bt_2') = b^{\lambda^{(2)}_{ac}} Q_r'^{ac}(t_1',t_2'), \qquad (33)$$

$$bt' = t, \qquad (34)$$

$$\phi^a_r(bt') = b^{\lambda^{(1)}_a}\phi_r'^a(t'), \qquad (35)$$

$$\ell_r(bt') = b^{\lambda_\ell}\ell_r'(t'), \qquad (36)$$

$$h_r(bt') = b^{\lambda_h}h_r'(t'). \qquad (37)$$

We then get

$$
S_K'[Q'] = (b^{2+\lambda^{(2)}_{ac}+\lambda^{(2)}_{\bar{a}\bar{c}}})\frac{1}{2}\sum_{r,r'} M_{rr'} \int_{\substack{\tau_0\le|t_1'-t_2'| \\ 0\le t_1',t_2'<\infty}} dt_1' dt_2' \times \sum_{a,c=0}^{1} Q_r'^{ac}(t_1',t_2') Q_{r'}'^{\bar{a}\bar{c}}(t_1',t_2'), \qquad (38)
$$

$$
S_{\text{HS}}'[Q',\phi'^0,\phi'^1] = (b^{2+\lambda^{(2)}_{ac}+\lambda^{(1)}_a+\lambda^{(1)}_c})\sum_r \int_{\substack{\tau_0\le|t_1'-t_2'| \\ 0\le t_1',t_2'<\infty}} dt_1' dt_2' \times \sum_{a,c=0}^{1} Q_r'^{ac}(t_1',t_2')\phi_r'^a(t_1')\phi_r'^c(t_2'). \qquad (39)
$$

Since the terms S_K and S_{HS} together represent the four-spin interaction that makes the model glassy, we demand that they both should be marginal under the RG. This leads to the conditions

$$0 = 2 + \lambda^{(2)}_{ac} + \lambda^{(2)}_{\bar{a}\bar{c}}, \qquad (40)$$

$$0 = 2 + \lambda^{(2)}_{ac} + \lambda^{(1)}_a + \lambda^{(1)}_c. \qquad (41)$$

The second condition can only be satisfied if $\lambda^{(2)}_{ac}$ is of the form $\lambda^{(2)}_{ac} = f(a)+f(c)$, where $f(a)\equiv -1-\lambda^{(1)}_a$. Inserting this form into the first condition, it yields $1+f(a)+f(\bar{a})=0$. At this point we still have freedom to pick among infinitely many possible solutions to this equation, each one of them defining a different RG transformation. We decide to choose the assignment $f(a)\equiv -a$, which leads to

$$\lambda^{(2)}_{ac} \equiv -a-c, \qquad (42)$$

$$\lambda^{(1)}_a \equiv a-1 = -\bar{a}. \qquad (43)$$

The choice of this particular solution is natural if we consider a reparametrization group (RpG) transformation,[38] associated with a reparametrization $s(t)$ of the time variables,

$$\tilde{Q}^{ac}_r(t_1',t_2') = \left(\frac{\partial s}{\partial t_1'}\right)^a \left(\frac{\partial s}{\partial t_2'}\right)^c Q^{ac}_r(s(t_1'),s(t_2')), \qquad (44)$$

where $a,c\in\{0,1\}$. For the special case of a rescaling of times, $s(t')=bt'$, Eq. (44) reduces to

$$Q^{ac}_r(bt_1',bt_2') = b^{-a-c}\tilde{Q}^{ac}_r(t_1',t_2'), \qquad (45)$$

which is completely analogous to Eq. (33) in the case $\lambda^{(2)}_{ac} = -a-c$.

For the source term, we demand that it should be marginal under the RG, and obtain the rescaling exponents

$$\lambda_\ell = -1 - \lambda_1^{(1)} = -1, \tag{46}$$

$$\lambda_h = -1 - \lambda_0^{(1)} = 0. \tag{47}$$

It can be checked that, besides S_K, $S_{\rm HS}$ and the source term, the boundary condition term $S_{\rm BC}$ is also marginal under the RG.

We now consider the effect of the RG transformation on the terms contained in $S_{\rm spin}[\phi^0,\phi^1]$. The time derivative term is not affected by the integration over fast modes, and the rescaling of times and fields introduces the following rescaling:

$$\frac{1}{\Gamma} \to \frac{1}{\Gamma'} = \frac{b^{1-\bar{0}-\bar{1}}}{b\Gamma} = \frac{1}{b\Gamma}. \tag{48}$$

If we now write

$$b = e^{\delta l}, \tag{49}$$

we get the RG equation

$$\frac{d\Gamma}{dl} = \Gamma. \tag{50}$$

Similarly we obtain

$$\frac{d\gamma_{ac}^{(2)}}{dl} = (a+c-1)\gamma_{ac}^{(2)}, \tag{51}$$

$$\frac{d\gamma^{(2p)}}{dl} = 0. \tag{52}$$

Finally, from Eq. (32), we find that the only term to receive a contribution from the integration over the fast degrees of freedom is the $g^{(4)}(t_1-t_2)$ term,

$$g^{(4)}(t_1-t_2) \to g'^{(4)}(t_1'-t_2') = b^{2-\bar{a}-\bar{c}-a-c}\big(g^{(4)}(bt_1'-bt_2') + \mathcal{C}_{\tau_0 \le |bt_1'-bt_2'| < b\tau_0}\big), \tag{53}$$

and we observe that for this term the rescaling prefactor evaluates to unity: $b^{2-\bar{a}-\bar{c}-a-c}=1$.

By examining the RG flow of Eqs. (50)–(53), we find the following fixed-point values:

$$\Gamma^* = 0,\infty,$$

$$\gamma_{00}^{*(2)} = 0,\infty,$$

$$\gamma_{01}^{*(2)} = \text{any number},$$

$$\gamma_{10}^{*(2)} = \text{any number},$$

$$\gamma_{11}^{*(2)} = 0,\infty,$$

$$\gamma^{*(2p)} = \text{any number} \quad \forall\, p \ge 2,$$

$$g^{*(4)}(t_1-t_2) = \mathcal{C}_{|t_1-t_2|<\tau_0}. \tag{54}$$

Since the RG flows of all parameters are uncoupled, the solutions above can be chosen independently for each parameter. The stability analysis around the fixed points shows that perturbations of Γ, $\gamma_{00}^{(2)}$, and $\gamma_{11}^{(2)}$ are *relevant* near $\Gamma^*=0$, $\gamma_{00}^{*(2)}=\infty$, and $\gamma_{11}^{*(2)}=0$, respectively, and are irrelevant near $\Gamma^*=\infty$, $\gamma_{00}^{*(2)}=0$, and $\gamma_{11}^{*(2)}=\infty$, respectively. It also shows that perturbations of $\gamma_{01}^{(2)}$, $\gamma_{10}^{(2)}$, and $\gamma^{(2p)}$ are marginal around any of their fixed points. Perturbations of $g^{(4)}(t_1-t_2)$ are always irrelevant. Therefore, for the set of initial conditions given by Eq. (26), and for almost any other set of initial conditions, the RG flows for Γ, $\gamma_{00}^{(2)}$, and $g^{(4)}(t_1-t_2)$ converge to their stable fixed points. However, the parameter $\gamma_{11}^{(2)}$ has a starting value, which is exactly at the unstable fixed point $\gamma_{11}^{*(2)}=0$, and stays there through the RG flow. Additionally, the parameters $\gamma_{01}^{(2)}$, $\gamma_{10}^{(2)}$, and $\gamma^{(2p)}$ do not flow at all, and stay at their initial values. In summary, the parameters of $S_{\rm spin}[\phi^0,\phi^1]$ flow to the fixed-point values,

$$\Gamma^* = \infty,$$

$$\gamma_{00}^{*(2)} = 0,$$

$$\gamma_{01}^{*(2)} = 2w_1,$$

$$\gamma_{10}^{*(2)} = 2w_1,$$

$$\gamma_{11}^{*(2)} = 0,$$

$$\gamma^{*(2p)} = 2pw_p \quad \forall\, p \ge 2,$$

$$g^{*(4)}(t_1-t_2) = \mathcal{C}_{|t_1-t_2|<\tau_0}. \tag{55}$$

As anticipated above, this result is the same for cutoff procedures (i) and (ii) [and in fact *for any other* possible initial value of $g^{(4)}(t_1-t_2)$].

The fact that Γ flows to infinity indicates that the derivative term does not appear in the fixed-point action. However, the states of the system at different times are still coupled by three other terms: $S_K[Q]$, $S_{\rm HS}[Q,\phi^0,\phi^1]$, and the term proportional to $g^{(4)}(t_1-t_2)$ in $S_{\rm spin}[\phi^0,\phi^1]$. We interpret this to indicate that, while the time derivative terms may be important for the short-time dynamics, when the short-time dynamics is integrated over and only the long-time dynamics remains, the coupling between different times is provided only by the terms associated to the spin-glass interactions. This is reminiscent of earlier mean-field calculations of the aging dynamics of spin glasses, in which the time derivative terms are negligible at long times, and the coupling between different times is also provided only by the spin-glass interaction terms.[4,5] In that context, the time derivative terms break the mean-field version of time reparametrization invariance, and therefore, this invariance is only valid for very long times when time derivative terms are negligible.[3–5,38]

By combining all the results for the RG flow for the various terms in the action, we find that the action converges to the fixed point,

$$S_{\mathrm{fp}}[Q] = F_\Omega + \frac{1}{2}\sum_{r,r'} M_{rr'} \int_{\substack{\tau_0 \le |t_1-t_2| \\ 0\le t_1,t_2<\infty}} dt_1 dt_2 \sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2) Q_{r'}^{\overline{ac}}(t_1,t_2)$$
$$- \ln \int \mathcal{D}\phi^0 \mathcal{D}\phi^1 \mathcal{D}\hat{\varphi} \exp\Bigg\{ i\sum_r \int_{\substack{\tau_0 \le |t_1-t_2| \\ 0\le t_1,t_2<\infty}} dt_1 dt_2 \sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2)\phi_r^a(t_1)\phi_r^c(t_2) - i\sum_r \int_0^\infty dt \Bigg(-2w_1 \phi_r^0(t)\phi_r^1(t)$$
$$- \sum_{p=2}^{\infty} 2p w_p \phi_r^0(t)(\phi_r^1(t))^{2p-1}\Bigg) - \frac{1}{2}\sum_{r,r'} K_{rr'} \int_{\substack{|t_1-t_2|<\tau_0 \\ 0\le t_1,t_2<\infty}} dt_1 dt_2 \sum_{a,c=0}^{1} \phi_r^a(t_1)\phi_r^c(t_2)\phi_{r'}^{\bar{a}}(t_1)\phi_{r'}^{\bar{c}}(t_2) + i\sum_r \hat{\varphi}_r\{\phi_r^1(0)-\varphi_r\}$$
$$+ \sum_r \int_0^\infty dt(\ell_r(t)\phi_r^1(t) + i h_r(t)\phi_r^0(t))\Bigg\}. \tag{56}$$

In this form the fixed-point action no longer contains the auxiliary fields $Q_r^{ac}(t_1,t_2)$ for times t_1 and t_2 such that $|t_1-t_2| < \tau_0$. We now reintroduce those auxiliary fields through the same Hubbard-Stratonovich transformation that was used to obtain Eq. (14), and obtain

$$S_{\mathrm{fp}}[Q] = F_\Omega + \frac{1}{2}\sum_{r,r'} M_{rr'} \int\int_0^\infty dt_1 dt_2 \sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2) Q_{r'}^{\overline{ac}}(t_1,t_2)$$
$$- \ln \int \mathcal{D}\phi^0 \mathcal{D}\phi^1 \mathcal{D}\hat{\varphi} \exp\Bigg\{ i\sum_r \int\int_0^\infty dt_1 dt_2 \sum_{a,c=0}^{1} Q_r^{ac}(t_1,t_2)\phi_r^a(t_1)\phi_r^c(t_2)$$
$$- i\sum_r \int_0^\infty dt\Bigg(-2w_1\phi_r^0(t)\phi_r^1(t) - \sum_{p=2}^{\infty} 2p w_p \phi_r^0(t)(\phi_r^1(t))^{2p-1}\Bigg) + i\sum_r \hat{\varphi}_r\{\phi_r^1(0)-\varphi_r\} + \sum_r \int_0^\infty dt(\ell_r(t)\phi_r^1(t) + i h_r(t)\phi_r^0(t))\Bigg\}. \tag{57}$$

V. REPARAMETRIZATION SYMMETRY

We are now finally ready to evaluate the effect of a reparametrization $t \to s(t)$ of the time variable on the fixed-point generating functional $\mathcal{Z}_{\mathrm{fp}}[\ell, h]$. We consider any smooth monotonous increasing function $s(t)$ satisfying the boundary conditions $s(0)=0$ and $s(\infty)=\infty$, which induces the following transformation of the sources:

$$\tilde{\ell}_r(t) = \left(\frac{\partial s}{\partial t}\right)\ell_r(s(t)), \tag{58}$$

$$\tilde{h}_r(t) = h_r(s(t)), \tag{59}$$

and compute the fixed-point disorder-averaged generating functional, evaluated at the transformed sources,

$$\mathcal{Z}_{\mathrm{fp}}[\{\tilde{\ell}_r(t),\tilde{h}_r(t)\}] = \int \mathcal{D}\tilde{Q} \exp\Bigg(-F_\Omega - \frac{1}{2}\sum_{r,r'} M_{rr'} \int\int_0^\infty dt_1 dt_2 \sum_{a,c=0}^{1} \tilde{Q}_r^{ac}(t_1,t_2)\tilde{Q}_{r'}^{\overline{ac}}(t_1,t_2)$$
$$+ \ln \int \mathcal{D}\psi^0 \mathcal{D}\psi^1 \mathcal{D}\hat{\varphi} \exp\Bigg\{ i\sum_r \int\int_0^\infty dt_1 dt_2 \sum_{a,c=0}^{1} \tilde{Q}_r^{ac}(t_1,t_2)\psi_r^a(t_1)\psi_r^c(t_2) - i\sum_r \int_0^\infty dt\Bigg(-2w_1\psi_r^0(t)\psi_r^1(t)$$
$$- \sum_{p=2}^{\infty} 2p w_p \psi_r^0(t)(\psi_r^1(t))^{2p-1}\Bigg) + i\sum_r \hat{\varphi}_r\{\psi_r^1(0)-\varphi_r\} + \sum_r \int_0^\infty dt(\tilde{\ell}_r(t)\psi_r^1(t) + i\tilde{h}_r(t)\psi_r^0(t))\Bigg\}\Bigg). \tag{60}$$

Here we have changed the name of the dummy variables from ϕ to ψ and from Q to $\tilde{Q}$ in the functional integral. We now perform the changes in variables $\psi_r^a(t) = (\frac{\partial s}{\partial t})^{\bar{a}}\phi_r^a(s(t))$ and $\tilde{Q}_r^{ac}(t_1,t_2) = (\frac{\partial s}{\partial t_1})^a(\frac{\partial s}{\partial t_2})^c Q_r^{ac}(s(t_1),s(t_2))$, i.e., the change in variables associated with the RpG transformation of Eq. (44), thus obtaining

$$
\begin{aligned}
\mathcal{Z}_{\rm fp}[\{\tilde{\ell}_r(t),\tilde{h}_r(t)\}] = \int \mathcal{D}Q\ \exp\Bigg(& -F_\Omega - \frac{1}{2}\sum_{r,r'} M_{rr'} \int\int_0^\infty dt_1 dt_2 \sum_{a,c=0}^{1} \left(\frac{\partial s}{\partial t_1}\right)^{a+\bar{a}} \left(\frac{\partial s}{\partial t_2}\right)^{c+\bar{c}} Q_r^{ac}(s(t_1),s(t_2)) Q_{r'}^{\overline{ac}}(s(t_1),s(t_2)) \\
& + \ln \int \mathcal{D}\phi^0 \mathcal{D}\phi^1 \mathcal{D}\hat{\varphi}\ \exp\Bigg\{ i\sum_r \int\int_0^\infty dt_1 dt_2 \sum_{a,c=0}^{1} \left(\frac{\partial s}{\partial t_1}\right)^{a+\bar{a}} \left(\frac{\partial s}{\partial t_2}\right)^{c+\bar{c}} Q_r^{ac}(s(t_1),s(t_2)) \phi_r^a(s(t_1)) \phi_r^c(s(t_2)) \\
& - i\sum_r \int_0^\infty dt \left(-2w_1 \left(\frac{\partial s}{\partial t}\right) \phi_r^0(s(t)) \phi_r^1(s(t)) - \sum_{p=2}^{\infty} 2pw_p \left(\frac{\partial s}{\partial t}\right) \phi_r^0(s(t)) (\phi_r^1(s(t)))^{2p-1} \right) + i\sum_r \hat{\varphi}_r\{\phi_r^1(s(0)) - \varphi_r\} \\
& + \sum_r \int_0^\infty dt \left\{ \left(\frac{\partial s}{\partial t}\right) \ell_r(s(t)) \phi_r^1(s(t)) + i h_r(s(t)) \left(\frac{\partial s}{\partial t}\right) \phi_r^0(s(t)) \right\} + \ln \mathcal{J}_1\left[\frac{\mathcal{D}(\psi^0\psi^1)}{\mathcal{D}(\phi^0\phi^1)}\right] \Bigg\} + \ln \mathcal{J}_2\left[\frac{\mathcal{D}\tilde{Q}}{\mathcal{D}Q}\right] \Bigg).
\end{aligned}
\tag{61}
$$

Here the symbol $\mathcal{J}_1[\frac{\mathcal{D}(\psi^0\psi^1)}{\mathcal{D}(\phi^0\phi^1)}]$ represents the Jacobian of the transformation from ϕ to ψ, and the symbol $\mathcal{J}_2[\frac{\mathcal{D}\tilde{Q}}{\mathcal{D}Q}]$ represents the Jacobian of the transformation from Q to $\tilde{Q}$. Since both transformations are linear transformations, the Jacobians only depend on the reparametrization $s(t)$, but they *do not* depend on the fields ϕ or Q, or the sources $\{\ell_r(t),h_r(t)\}$. For this reason, we will denote them as $\mathcal{J}_1\{s(t)\}$ and $\mathcal{J}_2\{s(t)\}$, respectively. Using the fact that $a+\bar{a}=1$ and $c+\bar{c}=1$, we find that the factor $(\frac{\partial s}{\partial t_1})^{a+\bar{a}}(\frac{\partial s}{\partial t_2})^{c+\bar{c}}$ is simply the Jacobian of the transformation from (t_1,t_2) to $(t_1',t_2')=(s(t_1),s(t_2))$, and therefore, we obtain

$$
\begin{aligned}
\mathcal{Z}_{\rm fp}[\{\tilde{\ell}_r(t),\tilde{h}_r(t)\}] = \int \mathcal{D}Q\ \exp\Bigg(& -F_\Omega - \frac{1}{2}\sum_{r,r'} M_{rr'} \int\int_0^\infty dt_1' dt_2' \sum_{a,c=0}^{1} Q_r^{ac}(t_1',t_2') Q_{r'}^{\overline{ac}}(t_1',t_2') + \ln \int \mathcal{D}\phi^0 \mathcal{D}\phi^1 \mathcal{D}\hat{\varphi} \\
& \times \exp\Bigg\{ i\sum_r \int\int_0^\infty dt_1' dt_2' \sum_{a,c=0}^{1} Q_r^{ac}(t_1',t_2') \phi_r^a(t_1') \phi_r^c(t_2') - i\sum_r \int_0^\infty dt' \Bigg(-2w_1 \phi_r^0(t') \phi_r^1(t') - \sum_{p=2}^{\infty} 2pw_p \phi_r^0(t') \\
& \times (\phi_r^1(t'))^{2p-1} \Bigg) + i\sum_r \hat{\varphi}_r\{\phi_r^1(0) - \varphi_r\} + \sum_r \int_0^\infty dt' (\ell_r(t')\phi_r^1(t') + i h_r(t')\phi_r^0(t')) + \ln \mathcal{J}_1\{s(t)\} \Bigg\} + \ln \mathcal{J}_2\{s(t)\} \Bigg) \\
= \mathcal{Z}_{\rm fp}[\{\ell_r(t),h_r(t)\}] \times \mathcal{J}_1\{s(t)\} \times \mathcal{J}_2\{s(t)\}.
\end{aligned}
\tag{62}
$$

Here we have used the boundary condition $s(0)=0$. We now consider the special case of zero sources, i.e., $\ell_r(t)=0$ and $h_r(t)=0$; in this case, the transformed sources are identical to the original ones, and we have the condition

$$\mathcal{Z}_{\rm fp}[\{0,0\}] = \mathcal{Z}_{\rm fp}[\{0,0\}] \times \mathcal{J}_1\{s(t)\} \times \mathcal{J}_2\{s(t)\}. \tag{63}$$

Since the generating functional is nonzero for zero sources (it is actually unity[39]), we immediately conclude that for *any* reparametrization $s(t)$, the product of the Jacobians is unity: $\mathcal{J}_1\{s(t)\} \times \mathcal{J}_2\{s(t)\} \equiv 1$. Thus we obtain, for *any* reparametrization $s(t)$, the identity

$$\mathcal{Z}_{\rm fp}[\{\tilde{\ell}_r(t),\tilde{h}_r(t)\}] = \mathcal{Z}_{\rm fp}[\{\ell_r(t),h_r(t)\}], \tag{64}$$

i.e., we have shown that the fixed-point generating functional is *invariant* under time reparametrization transformations.

VI. PHYSICAL CONSEQUENCES OF THE TIME REPARAMETRIZATION SYMMETRY

Since the renormalization group procedure described above involves integrating over all short-time-scale fluctuations, the fixed-point generating functional that we obtained controls the long-time dynamics of the model. The group of transformations associated with time reparametrizations is a continuous symmetry group for the fixed-point generating functional. This symmetry is broken by the actual dynamical correlations and responses observed in the system. As an example, let us consider the space-averaged two-time correlation $C(t,t_w) \equiv \frac{1}{N}\Sigma_r \langle \phi_r(t)\phi_r(t_w)\rangle$, where t_w is normally referred to as the "waiting time" and t as the "final time." If this correlation was actually invariant under time reparametrizations, we would have $C(t,t_w)=C(s(t),s(t_w))$ for any arbitrary increasing function $s(t)$ such that $s(0)=0$ and $s(\infty)=\infty$. The only possible way that this condition can be satisfied is if $C(t,t_w)=C_0$ (a constant). Since correlations in spin glasses actually do change with time, this implies that the reparametrization symmetry must be broken.

We have, therefore, the presence of a broken continuous symmetry group. Since no long-range interactions or gauge potentials are present, we should normally expect that a Goldstone theorem applies, giving rise to the presence of Goldstone modes (or soft modes) in the system.[40] For this reason, it has already been argued in Refs. 34–37 that Goldstone modes should be present in the nonequilibrium dynamics of spin glasses and possibly other glassy systems, and could in principle constitute the main source of fluctuations in the nonequilibrium dynamics of these systems. In other words, the presence of time reparametrization symmetry could account for a significant part of the dynamical heterogeneity effects observed in glassy systems.

In general, Goldstone modes are obtained from a continuous symmetry transformation by making it smoothly space dependent. For example, if the symmetry corresponds to spin rotations by any angle, to obtain a Goldstone mode the angle is chosen to be smoothly space dependent. In the present case, the continuous symmetry corresponds to reparametrizing the time variable $t \to s(t)$. In the uniform case, this leads to the symmetry transformation $C(t,t_w) \to \tilde{C}(t,t_w) = C(s(t),s(t_w))$. The Goldstone modes are obtained by choosing the time reparametrization to be smoothly space dependent, i.e., $t \to s_r(t)$ and $C_r(t,t_w) = C_0(s_r(t),s_r(t_w))$, where $C_0(t,t_w)$ is space independent.[35–37] Since the reparametrization is nonuniform, it is no longer a symmetry transformation for the system, but if the space variation is slow enough, the change in the action with respect to the value for a uniform two-time field is small. A possible (very simplified) physical interpretation of these Goldstone modes is that they are associated with "nonuniform slow relaxation:" if one considers different small regions in the system, for all regions the relaxation path is very nearly the same [as given by $C_0(t,t_w)$], but the rate at which each small region advances in its relaxation path can fluctuate from region to region.

Testing for the presence of fluctuations associated with this reparametrization symmetry has been performed in numerical simulations of both spin glasses and structural glasses. In spin glasses, one prediction that can be tested in simulations refers to the values of coarse grained *local* correlations $C_r(t,t_w) \equiv \frac{1}{n}\Sigma_{i\in B_r} s_i(t)s_i(t_w)$ and integrated responses $\chi_r(t,t_w) \equiv \int_{t_w}^{t} dt' \frac{1}{n}\Sigma_{i\in B_r}\frac{\delta\langle\phi_i(t)\rangle}{\delta h_i(t')}$, where B_r is a cubic coarse graining box containing n spins centered at the point r, in comparison to the *global* values $C(t,t_w)$ and $\chi(t,t_w)$ obtained by taking the averages over the whole sample. As explained in Refs. 35 and 36, the presence of Goldstone modes associated with time reparametrization symmetry would imply that the pairs (C_r,χ_r) should be concentrated predominantly along the parametric curve $\chi(C)$. It turns out that this is exactly what is observed in the results of numerical simulations in the three-dimensional (3D) Edwards-Anderson model.[35,36] Another testable prediction is that, if the global correlation $C(t,t_w)$ is only a function of the ratio t/t_w, i.e., $C(t,t_w) = \mathcal{C}(t/t_w)$, the probability distribution $\rho(C_r(t,t_w))$ for the values of the local coarse grained correlation $C_r(t,t_w)$ should collapse as a function of t_w, as long as t/t_w is held fixed. This has also been found to be the case in simulations in the 3D Edwards-Anderson model.[35,36] In Ref. 37 a more detailed study of the shape of the probability distributions for both the Edwards-Anderson model and a kinetically constrained model of glassiness was performed, with results that were consistent with the predictions derived from the presence of Goldstone modes in the system.

Another aspect of the results presented here that can be tested by comparison with numerical simulations in spin glasses is the fact that the symmetry is only exact for the fixed-point generating functional, i.e., in the limit $t \to \infty$. For the case of an exact continuous symmetry, one should expect that the presence of a true Goldstone mode (with zero mass) gives rise to spatial correlations that decay as power laws at long distances. However, at any finite time the symmetry is broken by small corrections to the action, which we can think of as small symmetry-breaking fields that go to zero at $t \to \infty$. As a consequence of the presence of these symmetry-breaking fields, the Goldstone modes now acquire a small mass, which should vanish in the $t \to \infty$ limit. In Ref. 36, the spatial correlation length $\xi(t,t_w)$ for fluctuations of the quantity $Q_r^{11}(t,t_w)$ was measured in large-scale long-time simulations in a 3D Edwards-Anderson model. For very large t, t_w, and t/t_w the time dependence of $\xi(t,t_w)$ was found to be consistent both with a form $\xi(t,t_w) \approx \ln(tt_w)$ or a form $\xi(t,t_w) \approx (tt_w)^a$, with $a \approx 0.04$. Both forms extrapolate (albeit slowly) to infinity at infinite times. This is suggestive and consistent with what is expected from the results of the present work, but the actual values of $\xi(t,t_w)$ are too small to make any firm statements about the $t \to \infty$ limit.

The present work only proves the presence of time reparametrization invariance in spin glasses. However, it is conceivable that the symmetry could extend to structural glasses, and there has already been some work in structural glasses, which has found suggestive evidence for its presence. In the case of structural glasses, there is another quantity which plays the role of local coarse grained correlation. It is defined[22] as $C_{\mathbf{r}}(t,t_w) = \frac{1}{N(B_{\mathbf{r}})}\Sigma_{\mathbf{r}_j(t_w)\in B_{\mathbf{r}}} \cos\{\mathbf{q}\cdot[\mathbf{r}_j(t) - \mathbf{r}_j(t_w)]\}$. Here $B_{\mathbf{r}}$ denotes a coarse graining box centered at the point $\mathbf{r}$ in the system, and the sums run over all of the $N(B_{\mathbf{r}})$ particles present in $B_{\mathbf{r}}$ at the waiting time t_w. The value of q is usually chosen to correspond to the main peak in the structure factor $S(q)$ of the system. Unlike in the 3D Edwards-Anderson model, in structural glasses the global correlation $C(t,t_w)$ is not a function of the ratio t/t_w. For this situation, the presence of the Goldstone mode associated with time reparametrization invariance implicates that the probability distribution $\rho(C_r(t,t_w))$ for the values of the local coarse grained correlation $C_r(t,t_w)$ should collapse as a function of t_w, as long as the global correlation $C(t,t_w)$ is held fixed.[36,37] This has been found to be the case, to a good approximation, in simulations in binary Lennard-Jones mixtures and binary Weeks-Chandler-Anderson mixtures.[22]

Confocal microscopy experiments in colloidal glasses[9–11] provide detailed data that include the positions of all colloidal particles in some subvolume of the sample at different times in the evolution of the system. These data can be analyzed in completely analogous ways to those used to analyze data from simulations in structural glasses. It remains an open question whether or not such analysis would provide further evidence in favor of the presence of time reparametrization symmetry.

VII. SUMMARY

In this work, we have presented a detailed proof of the presence of a symmetry under continuous reparametrizations of the time variable, for the long-time dynamics of a generic spin-glass model with two-spin interactions. No assumptions were made about the range of the interactions, therefore, the proof applies equally to short-range models, such as the Edwards-Anderson model, and to long-range models, such as the Sherrington-Kirkpatrick model. By performing a renormalization group procedure that exactly integrates over degrees of freedom associated with *short-time differences*, we have obtained the RG flow for the parameters in the action. We have found that the RG flow converges to a fixed-point generating functional, and we have explicitly written the form of this generating functional. Our main result is to have shown that the value of the fixed-point generating functional is left invariant by a transformation of the sources induced by a monotonous increasing but otherwise arbitrary reparametrization of the time variable.

The group of transformations associated with time reparametrizations is a continuous symmetry group for the fixed-point generating functional. This symmetry is broken by the actual dynamical correlations and responses observed in the system. In a situation like this, one would normally expect the presence of Goldstone modes. Indeed, it has been argued[34–37] that Goldstone modes associated with time reparametrization invariance should dominate the fluctuations in the nonequilibrium dynamics of these systems. Positive evidence for this statement has been found in simulations of the aging dynamics of the 3D Edwards-Anderson model.[35–37] Even simulations in systems without quenched disorder, such as kinetically constrained models of glassiness[37] and models of structural glasses,[22] show evidence in favor of the presence of this symmetry. Additionally, experimental tests for the presence of this symmetry in colloidal glasses can be provided by confocal microscopy measurements. Having proved the presence of time reparametrization symmetry, the present work opens the door for a more detailed analytical study of the symmetry itself, of the Goldstone modes probably associated with its presence, and more generally of the fluctuations ("dynamical heterogeneities") that are present in the slow dynamics of spin glasses and other glassy systems.

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